EVOLUTION OF (WARD–) TAKAHASHI RELATIONS AND HOW I USED THEM

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The story of (Ward–) Takahashi relations and their impact on physical theory is reviewed.

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Quantum field theories that describe realistically the fundamental interactions of elementary particles cannot be solved completely; mostly only approximate probes are available to us theorists. Consequently, when it happens that an exact result can be deduced, we are enormously pleased and cherish the derivation. That is why today we are happily commemorating the anniversary of one such exact observation – the (Ward–) Takahashi identity – which has elucidated the structure of quantum field theories and partially described their dynamical content, without recourse to any approximation scheme. This identity owes its final and general form to our colleague here in Edmonton, Yasushi Takahashi, and I shall review its evolution and successes over the years and also describe its impact on my research.

The story begins in 1950, when quantum field theory, more specifically spinor quantum electrodynamics, was being studied perturbatively and its divergences were being removed by renormalization. It soon became apparent that the multiplicative renormalization constant of the Fermion propagator $S(p)$ should coincide with that of the photon vertex function $\Gamma^\mu(p,q)$, and that this could be demonstrated if the propagator and vertex were related in some fashion. (The arguments of these functions are the four-momenta carried by the Fermions: a single p for the propagator; two values, p and q, for the vertex function, so that the photon momentum is $p-q$.)

A result that did the job was found by Ward, who recognized that a relation between the free expressions $S_0(p) = \frac{1}{i}(\not p - m)^{-1}$ and $\Gamma_0^\mu(p,q) = \gamma^\mu$, *viz.* the formula

$$\frac{\partial}{\partial p_\mu} S_0^{-1}(p) = i\gamma^\mu = i\Gamma_0^\mu(p,p)$$

also holds, order-by-order in perturbation theory, for the complete Fermion propagator and photon vertex at coincident arguments [1]. Ward's identity

$$\frac{\partial}{\partial p_\mu} S^{-1}(p) = i\Gamma^\mu(p,p) \tag{1}$$

directly implies the desired equality of renormalization constants. Moreover, immediately and independently, it was realized that Eq. (1) can also be used to give information about the photon vertex in the forward direction, for vanishing photon momentum,

$$\Gamma^\mu(p,q) \xrightarrow[p-q\to 0]{} -i\frac{\partial}{\partial p_\mu} S^{-1}(p)$$

With this information one can establish, without perturbation theory, exact low energy theorems for photon absorption, emission and scattering processes.

The first such threshold theorem was derived for the Compton amplitude, independently of Ward's result, by Thirring [2]; this was soon followed by the more refined analyses of Kroll and Ruderman, of Low, as well as of Gell-Mann and Goldberger, all of whom made explicit use of the Ward identity [3].

These successes brought with them new issues. First, one wondered if the identity could be established without recourse to the perturbation expansion or Feynman's diagrams. Second, one sought a proof for a conjectured [4], more general relation between the complete propagator and vertex at unequal Fermion momenta, which had been abstracted from the corresponding free-field formula.

$$S_0^{-1}(p) - S_0^{-1}(q) = i(p-q)_\mu \Gamma_0^\mu(p,q)$$

Finally, one needed to identify the intrinsic and general property of quantum field theory that is responsible for the validity of these relations, which evidently elucidate not only the structure of the theory as seen in its perturbative expansion, but also determine the infra-red (low-energy) limit of its dynamics.

The answer to all these questions is found in Takahashi's 1957 paper [5]. Its content is well summarized by the abstract, which is here reproduced.

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On the Generalized Ward Identity (*).

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Ward's identity (1) which shows the relation between the vertex operator with equal electron momenta and the electron propagator has been generalized for the case where the electron momenta are not equal. The generalized identity has not been rigorously proved, in spite of the fact that it is extensively used by many authors. The proof is given in this paper *without recourse to perturbation expansion or Feynman's diagram*. It is shown to be a consequence of the conservation of the current.

The paper provides a non-perturbative proof of the Takahashi relation

$$S^{-1}(p) - S^{-1}(q) = i(p-q)_\mu \Gamma^\mu(p,q) \tag{2}$$

thereby generalizing Ward's identity (1) away from the forward direction $p = q$. Also Takahashi's derivation clearly exposes the theoretical underpinnings of the argument.

His key observation is the formula

$$\frac{\partial}{\partial x^\mu}\Big(T j^\mu(x)\mathcal{O}(0)\Big) = [j^0(x), \mathcal{O}(0)]\delta(x^0) + T\Big(\frac{\partial}{\partial x^\mu} j^\mu(x)\mathcal{O}(0)\Big) \tag{3}$$

where T signifies time-ordering of the two local operators $j^\mu(x)$ and $\mathcal{O}(0)$. [In deriving (2), Takahashi let j^μ be the conserved electromagnetic current; he used two local operators

instead of the single $\mathcal{O}$ — two Fermi fields at different space-time points; and he worked in the Fourier transformed momentum space; but the essential ideas are already contained in (3).] The first term on the right, involving the δ-function in time (x^0), arises from differentiating the discontinuities of the T product and renders the coefficient commutator to be at equal times. The second term contains the divergence of the current operator, which vanishes if j^μ is a symmetry current, as in Takahashi's application. The equal-time commutator may be deduced with canonical commutation relations, when an explicit formula for j^0 is available — that is, one can use the correspondence principle to infer commutators from Poisson brackets. Another argument is based on the observation that the spatial volume integral of j^0, $Q = \int dV j^0$, generates by commutation some definite transformation on all dynamical variables. Indeed if j^μ corresponds to a conserved symmetry current, Q is the time-independent symmetry generator. If that transformation is explicitly known for the quantity $\mathcal{O}$ — call it $\Delta\mathcal{O}$ — than one has $i[Q, \mathcal{O}(0)] = \Delta\mathcal{O}(0)$ and it is plausible to suppose that

$$i[j^0(x), \mathcal{O}(0)]\delta(x^0) = \Delta\mathcal{O}(0)\delta(x^0)\delta(V) \tag{4}$$

Even if no symmetry is present so the current is not conserved, formula (3) can still provide information if an argument can be given for accepting (4), and if the divergence of the current, $\partial_\mu j^\mu$, is an operator with known properties (in which case one says that j^μ is *partially conserved*). Takahashi's analysis was generalized to the case of T products with several currents by Kazes [6].

The derivation establishes that the foundations of the (Ward–) Takahashi identity and its generalizations rest on the broad basis of equal-time commutators, transformations and symmetries, (rather than on the more restrictive conditions of gauge-invariance [7]), and demonstrates that the dynamical consequences — the low-energy theorems [2,3] — are truly non-perturbative.

The roles of these identities in quantum electrodynamics were played out by the 1960's. But just then quantum field theory research advanced in a new direction, which became accessible thanks to the information contained in the generalized Takahashi relations.

At that time, physicists possessed neither a field theoretical model for fundamental interactions (other than electromagnetism) nor could they solve any proposed candidate theory to see whether it is viable. But it was appreciated that matrix elements of various vector and axial-vector currents, corresponding to various internal groups, govern experimentally accessible processes and carry important information about fundamental interactions. Absence of a realistic theoretical model and of an effective calculational method stymied progress: the values of these matrix elements could not be determined *a priori*.

The breakthrough came when Gell-Mann postulated his "current algebra," giving explicit form to the commutators (4). For j^0 he took the time components of the physically

interesting currents; for $\mathcal{O}$, the full four-vector or the axial four-vector current; and for $\Delta\mathcal{O}$, the expected transformation that follows from the relevant group structure: *i.e.* $\Delta\mathcal{O}$ is the infinitesimal group rotation of $\mathcal{O}$. Also a definite form for the divergences of these currents was assumed: conserved for the approximate vector symmetries; partially conserved or spontaneously broken for the axial-vector currents [8]. With this information, the right side of the Takahashi identity (3) is explicitly known and serves to constrain the left side – *i.e.* the matrix element of the physically interesting current.

Thereupon followed an explosion of activity that resulted in an understanding of spontaneous symmetry breaking and of the Nambu–Goldstone as well as the Anderson–Higgs phenomena; in low energy theorems, principally for pseudoscalar mesons that couple to axial-vector currents; in sum rules for scattering processes; and in a variety of other calculations that mostly agreed well with experimental observation [9]. These successes also provided an enthusiastic vote of confidence for operator quantum field theory, since current algebra and the Takahashi identities do not fit into any other frame.

This was the time that I began physics research. As a graduate student, I had been frustrated by the absence of a quantum field theoretic description for non-electromagnetic fundamental processes, so I was delighted to learn about Gell-Mann's suggestion and the subsequent progress in current algebra. Earlier, while studying quantum mechanics, I was very much impressed that by postulating a largely model-independent equal-time commutator algebra for position operators of the electron, $i\left[\frac{d}{dt}r^i, r^j\right] = \frac{1}{m}\delta^{ij}$ ($m =$ electron mass), one could derive the Thomas–Reiche–Kuhn sum rule for transition probabilities, without solving any dynamical equations or even adopting any particular interaction model. I recognized in current algebra a quantum field theoretic analog of the successful quantum mechanical program for sum rules [10] and decided to enter into that research.

But the subject had reached maturity, with little room for new discoveries. However, with collaborators, we managed to find yet another set of low energy theorems: threshold relations for gravitons [11] that are completely analogous to the ones for photons [2,3]. Since graviton emission is not yet experimentally accessible the results are academic; but they are interesting in that novel Takahashi relations, based on Poincaré symmetry and energy-momentum conservation, are used to analyze processes for which no quantum theory exists: quantum gravity has not yet been constructed! This demonstrates vividly the power of these relations in making physical predictions.

The success of current algebra was marred by uncertainty over the actual existence of a field theoretical model in which the postulated relations are realized and the theorems about dynamics are true. Indeed a potential obstacle was recognized from the beginning of the (Ward–) Takahashi/current algebra program. It was known that local equal time commutators frequently differ from the respective Poisson brackets; the correspondence principle fails. Specifically when in (4) $\mathcal{O}$ is the spatial component of the current j^μ, there are fur-

ther contributions, proportional to spatial derivatives of δ-functions. These terms, called *Schwinger terms*, were exhibited by Takahashi's compatriots, Goto and Imamura [12], but in fact they were discovered by Jordan in the 1930's [13]. Lack of *a priori* information about the Schwinger terms prevents evaluation of (4) and of the right side in (3). However, it was also noted that physical amplitudes differ from time-ordered products by further local terms, called *seagulls*, and in explicit examples, like the Compton scattering amplitude for scalar electrodynamics, one verified that the divergence of the seagulls cancels exactly the Schwinger terms, leaving the "naive" result, obtained by ignoring the problem altogether. Taking an optimistic position, Feynman conjectured that this cancellation is generally valid, and that one should work with unmodified Takahashi identities.

While such pragmatism allowed calculations to proceed fearlessly, it did not satisfy me, especially since there remained discrepancies between some current algebra predictions and experimental facts. At CERN, where I came as a visiting researcher, Bell emphasized especially that the current algebra analysis of neutral pion decay, performed by his colleages Sutherland and Veltman, leads to the unacceptable conclusion that a massless pion does not decay into two photons. But a non-vanishing decay time had been measured and its magnitude could not be attributed to the small pion mass. Bell stressed that the subject of current algebra must not be closed without resolving this puzzle.

I decided to elucidate this question, but at first made little progress because the problematic conclusion presented itself in very immediate form, being a straightforward consequence of Takahashi identities. One could consider, in the Heisenberg picture, the vacuum-two photon matrix element of the axial-vector current j_5^μ, to which the pion couples,

$$T^\mu(z) = \langle \gamma\gamma | j_5^\mu(z) | 0 \rangle$$

T^μ must be invariant against gauge transformations on the two photons. Also in the limit of massless pions, j_5^μ was taken to be conserved thanks to chiral symmetry, which was believed to hold in that limit. Consequently, T^μ should be transverse. Alternatively, one could work in the electromagnetic interaction picture and consider the vacuum amplitude of three currents: two electromagnetic currents to which the two photons couple and j_5^μ

$$T^{\alpha\beta\mu}(x,y,z) = e^2 \langle 0 | T j^\alpha(x) j^\beta(y) j_5^\mu(z) | 0 \rangle$$

(On photon mass shell $T^{\alpha\beta\mu}$coincides with T^μ.) Now the conservation of all three currents (which satisfy free field equations in the interaction picture) together with the current algebra assumption of vanishing commutators leads to transversality of $T^{\alpha\beta\mu}$ in all its three indices. Either of these two generalized Takahashi identities prohibits massless pion decay, and it was difficult to see how the conclusion could be evaded.

A civilized activity at CERN consists of taking an afternoon drink at the cafeteria. Bell and I frequently went there, together with other people who joined us for discussions. On

one occasion, Steinberger was at the table and asked about our current interests. When we described to him our puzzlement about $\pi^0 \to 2\gamma$, he expressed amazement that theorists should still be pursuing a process that he, an experimentalist, had calculated twenty years earlier, finding excellent agreement with experiment [14].

There at the cafeteria table, Bell and I realized that Steinberger's calculation would be identical to the one performed in the dynamical framework of the σ-model, which was constructed to realize current algebra explicitly. We reasoned that within the σ-model, we could satisfy the current algebraic assumptions of Sutherland–Veltman and also obtain good experimental agreement in view of Steinberger's result, thereby resolving the $\pi^0 \to 2\gamma$ puzzle.

Guided by Steinberger's paper (at that time, we were not familiar with similar work by Takahashi's compatriots Fukuda and Miyamoto and only dimly aware of Schwinger's position-space reanalysis of the Fukuda–Miyamoto–Steinberger momentum-space calculation [15]) we quickly established that the lowest order amplitude describing correlations between the three currents appearing in the problem is given to lowest (one-loop) perturbative order by the now famous triangle graph and that the explicit result does not obey the expected Takahashi identities: the single index, vacuum-two photon matrix element of j_5^μ, $T^\mu(z)$, is not transverse when photon gauge invariance is enforced; the three-index, three-current amplitude $T^{\alpha\beta\mu}(x,y,z)$ cannot be transverse in all three indices — transversality is possible in at most two. Since the Sutherland–Veltman argument relies on transverality of these amplitudes, the undesirable conclusion does not hold [16].

From this calculation, Takahashi identities took on a new life, because in modified form, they summarize the violation of the correspondence principle by recording the "anomalies" that violate the naive, expected formulas. In the case of the vacuum-two photon matrix element of j_5^μ, $T^\mu(z)$, the anomaly resides in the divergence of the axial vector current: rather than being conserved it satisfies

$$\partial_\mu j_5^\mu \propto {}^*F^{\mu\nu} F_{\mu\nu} \ ,$$

where $F_{\mu\nu}$ is the electromagnetic field strength and ${}^*F^{\mu\nu}$ is its dual [17]. On the other hand, when considering the three current vacuum correlator $T^{\alpha\beta\mu}(x,y,z)$, all currents are conserved — they are free in the interaction picture — but the anomaly resides in the commutator algebra: rather than vanishing commutators, one has an unexpected Schwinger term

$$[j^0(x), j_5^0(y)]\delta(x^0 - y^0) \propto B^i(y)\delta(x^0 - y^0)\partial_i \delta^3(\mathbf{x} - \mathbf{y})$$

which does not cancel with the corresponding seagull, leading to the anomalous Takahashi relation (here B^i is the magnetic field). Consequently Feynman's conjecture is violated [18]. Anomalous divergences and anomalous commutators are different sides of the same coin, and they are unified by giving an altered form to the Takahashi relation.

After the above discovery resolved the pion decay puzzle, there followed a search for departures from the correspondence principle in other (Ward–) Takahashi relations. These were found in the so-called trace identities, which involve the trace of the energy-momentum tensor and result from scale/conformal invariance. Just as in the previous example of chiral symmetry, scale/conformal symmetry in quantum field theory also acquires an anomaly [19], and the modified Takahashi identities for this case were recognized as expressions of the renormalization group — an analysis of scaling violation in quantum field theory that had been carried out many years before [20].

Understanding the departures from naive, canonical reasoning came more or less at the same time as the construction of the "standard" quantum field theoretic Yang–Mills model of all fundamental interactions (excluding gravity). Within this model the (Ward–) Takahashi identities — for the non-Abelian context they became known as Slavnov–Taylor identities — again aided in the renormalization procedure, which is successful only if the identities are free of anomalies. This could be achieved provided quarks and leptons are precisely balanced in number and charge. Just such a balance is observed, which is striking evidence that not only quantum field theorists, but also Nature knows about the subtleties of the Takahashi identities.

On the other hand, identities that play no role in renormalization do contain contributions that violate the correspondence principle and lead to a variety of physical effects in the standard model [21]. For example unwanted symmetries, like scale invariance and the pion-decay forbidding chiral invariance, disappear from the quantum field theory, though they are present in the classical theory, before quantization. Another instance is an unexpected and unwanted source of CP violation, caused by quantum tunneling. Perhaps most intriguing is 't Hooft's demonstration that the anomaly in the Takahashi identity for the baryon number current allows baryons to decay, but fortunately at a sufficiently slow rate so that the threat to our physical world is negligible [22].

In addition to the physics that has emerged from the study of (Ward–) Takahashi relations, there also arose a very fruitful exchange of ideas with mathematics, which is still flourishing. This happened because it was noticed that the entities that characterize the modifications to the Takahashi identities, be they anomalous divergences of currents or anomalous commutators, possess mathematical significance for geometry and topology [21]. The subsequent mathematics-physics collaboration is most vividly seen these days in the string program; if it lives up to the promises that its supporters proclaim, it will surely provide a future arena for new roles to be played by (Ward–) Takahashi relations.

I wish Yasushi Takahashi a long life, so he can witness and enjoy the further development of his identities, which evolved from a simple observation in a corner of physics into a concept that is relevant to much of the formalism that we use in describing Nature at its fundamental workings.

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